Topological Groups

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Abstract

These are notes to accompany an introductory level talk on Topological Groups given on 19 October 2008 at Queens College. The goal is to first define a topological group and examine some of the properties which follow immediately from the definition. These will be used to elucidate some facts about the the topological structure of these objects. We will then examine the algebraic properties, by defining (normal) subgroups of topological groups, factor groups of topological groups (though much of the construction of factor groups, which is long and cumbersome, is omitted), and homomorphisms and isomorphisms of topological groups, and proving the topological groups analogue of a familiar theorem from Group Theory. Lastly we will introduce the notion of local isomorphy, and state without proof a theorem which helps to illustrate the significance of local isomorphism classes.

It should be noted that whenever we wish to deemphasize the topological properties of our group (or other structure) we will refer to it as an *algebraic* group, to mean that we are only concerned with its algebraic structure.

Most of the content of and cause for this talk either from Doctor Itzkowitz's Point Set Topology course, and so a special thanks is due to him. Lev Semenovich Pontryagin's work Topological Groups was also relied upon heavily while putting together these notes and some of the proofs given are his.

1 Basic Definitions and Properties

A Topological Group $(G, \tau, \cdot)$ is a group $(G, \cdot)$ equipped with a topology τ such that the function

$$f : G \times G \to G(g_1, g_2) \mapsto g_1 g_2^{-1}$$

is jointly continuous. For simplicity we will generally just refer to a topological group $(G, \tau, \cdot)$ as G.

A subset U of a topological group G is called symmetric if $u \in U \Rightarrow u^{-1} \in U$

Examples of Topological Groups

$(\Re, +)$ with the usual (Euclidean) topology is a topological group.

$(\Re^+, \cdot)$ with the usual (Euclidean) topology is a topological group.

All Lie Groups are topological groups. (A Lie Group is a finite dimensional smooth manifold such that group multiplication and inversion are smooth maps)

One such Lie Group is

$$GL_2(\Re) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \det(A) \neq 0 \right\}$$

where the group operation is matrix multiplication.

Elementary Properties of Topological Groups

Functions of the following forms are homeomorphisms

1. left translations $l_a(x) = ax$
2. right translations $r_a(x) = xa$
3. inversions $g(x) = x^{-1}$
4. inner automorphisms $\varphi_a(x) = a^{-1}xa$

Corollaries

1. if $F \subseteq G$ is closed and a is any element of G, then Fa aF and F^{-1} are all closed as well.
2. if $U \subseteq G$ is open, $P \subseteq G$ is any subset, and a is any element of G, then UP PU U^{-1} aU and Ua are all open as well.
3. if A,$B \subseteq G$ are both compact, then AB is compact.

If $e \in G$ is the identity of a topological group, then for any neighborhood U of e there exists a symmetric neighborhood W of e such that $W \subseteq U$. Equivalently, there is a neighborhood base at the identity consisting of symmetric neighborhoods.

Let U be a neighborhood of the identity, e. Then U^{-1} must also be a neighborhood of e, since taking inverses is a homeomorphism and since $e^{-1} = e$. Then $W = U \cap U^{-1}$ is also a neighborhood of e. W is clearly symmetric, since $g \in U \Rightarrow g^{-1} \in U^{-1}$ and $g \in U^{-1} \Rightarrow g^{-1} \in U$.

Each neighborhood U of e contains a symmetric neighborhood W of e such that $W^2 \subseteq U$

Let U be a neighborhood of the identity, e. Then by the continuity of multiplication, since $ee = e$ we can find neighborhoods W_1 and W_2 of e such that $W_1 W_2 \subseteq U$. Then let $W = W_1 \cap W_2$. W is clearly symmetric, and $W \subseteq W_1$ and $W \subseteq W_2$. So $W^2 \subseteq W_1 W_2 \subseteq U$

2 Topological Structure

A topological group is homogenous. That is, $\forall\ g_1, g_2 \in G$, $\exists f$ such that f is a homeomorphism and $f(g_1) = g_2$. One such homeomorphism is given by $l_{g_2 g_1^{-1}}$ Therefore, when concerned with local properties, it is sufficient to prove that they hold in neighborhoods of the identity to see that they hold in neighborhoods of every point in the space.

Topological Groups are Regular. By regular we mean that there exist neighborhood bases for every point in G consisting of closed neighborhoods.

Let U be a neighborhood of e. Then $\exists$ a neighborhood W of e such that $W^2 \subseteq U$. For any $x \in \overline{W}$, every neighborhood of x meets W and xW is a neighborhood of x. Therefore we can find $y, z \in W$ such that $xy = z \Rightarrow x = zy^{-1} \in WW^{-1} \subseteq U \Rightarrow \overline{W} \subseteq U$ and so the topological space is regular.

For a topological group G, the following are equivalent

1. G is T_0
2. G is T_1
3. G is T_2 (Hausdorff)
4. $\bigcap_{U \in \mathcal{U}} U = \{e\}$ where $\mathcal{U}$ is a neighborhood base for the identity, e.

(4 $\Rightarrow$ 3) Let $x \in G$, $x \neq e$. Then $\exists$ a neighborhood V of e such that $x \notin V$. Let W be a symmetric neighborhood of e such that $W^2 \subseteq V$. Then $xW \cap W = \emptyset$. By the homogeneity of G, we can therefore seperate any two points by neighborhoods in this fashion, and the space is T_2.

3 $\Rightarrow$ 2 $\Rightarrow$ 1 is clear.

(1 $\Rightarrow$ 4) Let $x \in G$, $x \neq e$. Then either x fails to be in some neighborhood of e or e fails to be in some neighborhood of x. Suppose the latter.

Then, $\exists$ a neighborhood U of x such that $e \notin U$. Since multiplication is continuous, $\exists$ a neighborhood V of e such that $Vx \subseteq U$. Now let W be a symmetric neighborhood of e such that $W^2 \subseteq V$. Then $W \cap Wx = \emptyset$ since if $\exists y \in W \cap Wx$, then $\exists w \in W$ such that $y = wx \Rightarrow e = y^{-1}wx \in W^2x \subseteq Vx \subseteq U$ and $e \notin U$

3 Algebraic Structure

A subgroup H of a topological group G is an algebraic subgroup H which is closed in the topology on G. H is normal if it is normal as an algebraic subgroup.

If H is a normal algebraic subgroup of G, then $\overline{H}$ is a normal topological subgroup of G.

1. $\overline{H}$ is a subgroup: Let a,$b \in \overline{H}$ and let W be a neighborhood of ab^{-1}. Then we can choose neighborhoods U of a and V of b such that $UV^{-1} \subseteq W$. Also, $\exists x, y \in H$ such that $x \in U$ and $y \in V$. Then $xy^{-1} \in W$ and since H is an algebraic subgroup of G, $xy^{-1} \in H$ This implies that $ab^{-1} \in \overline{H}$ and therefore $\overline{H}$ is also a subgroup of G.

2. $\overline{H}$ is normal: Let $a \in \overline{H}$ and $c \in G$. Let V be a neighborhood of $c^{-1}ac$. Then since this is an inner automorphism, $\exists$ a neighborhood U of a such that $c^{-1}Uc \subseteq V$. Since $a \in \overline{H} \exists x \in U \cap H$ so $c^{-1}xc \in U$ and $c^{-1}xc \in H$. Therefore $c^{-1}ac \in \overline{H}$ and $\overline{H}$ is normal.

If H is an algebraic subgroup of G and H is open, then $H = \overline{H}$.

Let $x \in \overline{H}$. Then $xH \cap H \neq \emptyset$ so $x \in HH^{-1} = H$

We can construct a topological group analogue of the notion of a factor group as follows. Denote by G/H the collection of all right cosets of the subgroup H in G. Let Σ be a base for the topology on G. $\forall\, U \in \Sigma$ define $U' = \{Hx \colon x \in U\}$ Then $\Sigma' = \{U' \colon U \in \Sigma\}$ gives a complete system of neighborhoods for a new topology on G/H and G/H can be shown to be a topological group with this topology.

An <u>isomorphism of topological groups</u> is a group isomorphism which is homeomorphic. A <u>homomorphism of topological groups</u> is a group homomorphism which is continuous. The following theorem however will show that it is more natural to consider open homomorphisms (that is, homomorphisms of topological groups which map open sets to open sets) as the topological group extension of group homomorphisms.

Let G and G' be T_0 topological groups. Let $g \colon G \to G'$ be an open homomorphism with $\ker(f) = N$. Then N is a normal topological subgroup of G and $G/N \simeq G'$

From group theory we know that N is an algebraic subgroup of G and since N is the continuous inverse image of a single point, e', N will be closed and therefore a topological subgroup of G.

We also know from group theory that if we choose some $x' \in G'$ and let $X = g^{-1}(x')$ that X is a coset of N in G. Then we can define $f\colon G' \to G/N$ by $f(x') = X$ and f will be an algebraic isomorphism. Then it only remains to show that f is a homeomorphism.

Let $a' \in G'$ and let $f(a') = A$. Let U' be a neighborhood of A in G/N. From our earlier construction we know that $U' = \{Nx\colon x \in U\}$ where U is some fixed neighborhood in G. Choose a point $a \in U$ such that $A = Na$. Then $g(a) = a'$ and since g is open, $\exists$ a neighborhood V' of a' such that $V' \subseteq g(U)$. Therefore $f(V') \subseteq U'$. To see this let $y' \in V'$. Then $\exists y \in U$, $g(y) = y'$ Then $f(y') = Ny \in U'$ and so f is continuous. Were g not open f would fail to be continous and we would already fail to have an isomorphism of topological groups.

It remains to show that f^{-1} is continous. Let $A = Na \in G/N$ and let $f^{-1}(A) = a'$. Then let U' be a neighborhood of a'. $g(a) = a'$ and g is continous so $\exists$ a neighborhood V of a such that $g(v) \subseteq U'$. If we now define $V' = \{Nx\colon x \in V\}$ then $f^{-1}(V') \subseteq U'$ and f^{-1} is continuous.

Two topological groups G and G' are said to be locally isomorphic if $\exists$ neighborhoods U of e and U' of e' and a homeomorphism $f\colon U \to U'$ satisfying that

1. $x, y, xy \in U \Rightarrow f(xy) = f(x)f(y)$
2. $x', y', x'y' \in U' \Rightarrow f^{-1}(x'y') = f^{-1}(x')f^{-1}(y')$

Let G and G' be topological groups generated by arbitrary neighborhoods of their identities. (If G and G' are connected then this will follow). Then if G is locally isomorphic to G', $\exists$ a topological group H with discrete normal subgroups N and N' such that $G \simeq H/N$ and $G' \simeq H/N'$.